Joe D’Antoni offered a string figure challenge concerning a double walled diamond figure (DWD) "stone money" as discussed on Martin Probert's string figure web site. D’Antoni posed the following questions to be asked in the analysis of the string figure and its look-alikes.

1. How many look-alike string figures are possible?
2. How many of these string figures are actually stable when strings are held under tension?
3. How many of the remaining string figures can be made from one string loop? Alternately, we can ask, "When dissolved, how many figures will not become knotted?"

The specific challenge for the "stone money figure" that D'antoni posited was:
How many stable double-walled string figures are possible? (Note: (he says) I am not asking you to answer Question 3).
"stone money" from Probert's web site is below.
What is the difference in the two figures illustrated below?


Fig. 1 - Stone Money
Caroline Islands 1902 (Jayne)


Fig. 2 - Star
Guyana c. 1924 (Roth)

The labelling indicates the points at which the figures differ. E, F, G and H of Stone Money (see our fig. 1; Jayne's fig. 359 is incorrect) differ from e, f, g and h of Star (see our fig. 2; Roth’s fig. 267 is incorrect). Such similar looking figures will be called look-alikes.

I decided to try to answer his question(s).
The figures I knew which produced a look-alike figure were "catch the sun" and an artificial system of my devising for my high school students. Once back in the 80's, I shaped up for jury duty in new york city civil court. It wasn't that I was eager for civil court, criminal being much more exciting in my mind; but being an early arriver I was rewarded. Only the obsessively early were sent over to civil court from the main shape-up room.

While I sat there in the holding pen I played with my strings. At that time, shortly after I had begun to teach strings, I couldn’t get enough of discovering new figures, and welcomed the idle hours of play while waiting in the jury room.

A young man approached me and watched for a while. He had a Spanish accent and looked Indian to me. He asked me about what I did with the strings and I explained my interests. He said that he was an Aymara from Bolivia and that he had learned how to "catch the sun" while a little boy. I said show me how and he did.

## Catch the Sun

OA, OB, ldna, rdna

1) all four fingers down into thumb loop from above
2) near thumb string is thrown over to far little finger side of hand, hands back to position with fingers pointing up. Resulting hand loop falls down to back of writsts.
3) thumb up through back of hand loop and index loop
4) thumbs pull down far index strings, pick up far little finger strings and hold them tight to the palm with the thumbs
5) throw back-of-hand loops off hands
6) open index and middle fingers to loose string
7) pull hands apart with thumbs and little fingers to "catch the sun"

Or:
OA, OB, ldna, rdna

1) index loops $-1 / 2$
2) 5 loops down through 2 loops
3) 1 loops up through 2 loops
4) drop 2 loops and extend

These figures can be rotated around horizontal and vertical axes as well as rotated 180 degrees around a center point. These will represent new configurations of crossing patterns when viewed by the extender of the figure.

Joe D'Antoni showed me another way to form look-alikes that I didn't know.
D'Antoni One

1. Opening A.
2. R1, over R2 loop, picks up R5n. R3, over palm string and R2 loop, picks up lower R1f. Release 1 loops. R1, over R2 loop, and from below removes R3 loop.
3. R5, over R2 loop, picks up R1f. R2, over palm string and R2 loop, picks up lower R5n. Release 5 loops. R5, from below, removes upper R2 loop.
4. Release R2 loop.
5. R2, from above, removes L2 loop.
6. R1 and R5, from below, remove R2 loop.
7. Navajo R1 and R5.
8. R1, from above, removes L1 loop. R5, from above, removes L5 loop.
9. Turn $R$ hand so fingers point o you. L1, from below (proximally), removes $R 5$ loops.
10. 345 hooks down lower 1 f against palm.
11. Caroline Extension. A bit of string tugging is required.

He also sent along this further information.
In the literature I located two other DWD's that share the odd features shown in the scan of my figure. Their method of construction is different from my own, and yields a distinct figure. Here is my original figure.


Below is a figure from Roth, String Figures, Tricks, and Puzzles of the Guiana Indians.


Fig. 267.-Star. (Wapishana. etc.)

Below is a figure from Lutz, String Figures from the Patomana Indians of British Guiana.


Fig. 11. A Star.

Roth and Lutz offer identical methods for their figures as follows:

1. Position 1.
2. R2 hooks up Lp, rotating toward you and up.
3. L2, from above, into R2 loop, and hooks up Rp, rotating toward you and up.
4. 1 loop, from below, through 2 loop; reset onto 1.
5. 2 loop, from below, through 1 loop; reset onto 2.
6. 5 loop, from below, through 2 loop; reset onto 5.
7. Release 2 loop.

This reminds me of your work on loop transfer and circle notation. Unfortunately, their method does not result in what they illustrated (two crossings are different). Instead, you get the following:


I have discovered how to make Lutz/Roth illustration, as follows:

1. Position 1.
2. $R 2$ hooks up Lp, rotating away from you and up.
3. L2, from above, into R2 loop, and hooks up Rp, rotating away from you and up.
4. 5 loop, from below, through 2 loop; reset onto 5.
5. 2 loop, from below, through 5 loop; reset onto 2.
6. 1 loop, from below, through 2 loop; reset onto 1 .
7. Release 2 loop.

In addition to my original figure, I produced one more DWD figure by adding one step (in bold) to my original instructions. It differs from my original figure by just one crossing.

D’Antoni Two

1. Opening A.
2. R1, over R2 loop, picks up R5n. R3, over palm string and R2 loop, picks up lower R1f. Release 1 loops. R1, over R2 loop, and from below removes R3 loop.
3. R5, over R2 loop, picks up R1f. R2, over palm string and R2 loop, picks up lower R5n. Release 5 loops. R5, from below, removes upper R2 loop.
4. Release R2 loop.

L2 loop <<.
5. R2, from above, removes L2 loop.
6. R1 and R5, from below, remove R2 loop.
7. Navajo R1 and R5.
8. R1, from above, removes L1 loop. R5, from above, removes L5 loop.
9. Turn $R$ hand so fingers point o you. L1, from below (proximally), removes $R 5$ loops.
10. 345 hooks down lower 1 f against palm.
11. Caroline Extension. A bit of string tugging is required.


The following illustration Joe gave me shows the issues more clearly. Given the figure:


1. How many look-alike string figures are possible?
2. How many of these string figures are actually stable when strings are held under tension?
3. How many of the remaining string figures can be made from one string loop? Alternately, we can ask, "When dissolved, how many figures will not become knotted?"

## Answer to Question 1

There are nine string crossings: the upper frame line is crossed four times, the lower frame line is crossed four times, and one crossing in the middle of the figure. Strand A and strand B can only cross in two ways: strand A over strand B, or strand B over strand A. Thus 2 to the 9th, or 512 look-alike string figures can be drawn.

## Answer to Question 2

Refer to the two-diamond figure, and consider the two string crossings at the upper left. Four different arrangements are possible. See the over-under image below:


The diamond wall will collapse if the string crosses over-over or, under-under the frame line. That leaves only two stable configurations. The same argument is valid for other places in the figure where a frame line is crossed. Thus only 2 to the 5th, or 32 of our original figures will be stable string figures.

Answer to Question 3
It is fairly obvious that the five string crossings in our original figure can be removed by simply untwisting them. The same holds true for any of the 32 figures. So, out of our original 512 potential string figures there are only 32 unique look-alike two-diamond string figures that can be made from a single string loop.

I will not address the question of how to make these 32 string figures.

Now, I would like to challenge the string group. Double-walled diamond string figures are ubiquitous. For example see "stone money"

How many stable double-walled string figures are possible? (Note: I am not asking you to answer Question 3).

Armed with this information I devised an approach to answering the questions for the DWD (double walled diamond) figures. First I prepared a crossing pattern grid to replicate and sketch in the different figures.


There are 16 string crossing as numbered (1-10 are "outside" crossings, 11-16 are "inside crossings").

Therefore there are 2 to the $16^{\text {th }}$ or 66,536 possible different crossing patterns for the double DWD.

The following is the "Catch the Sun" figure formed with ldna, using my frame.


My encoding of the figure (and I will use this procedure throughout this paper) is TT (top transverse) uoou, BT (bottom transverse) ouuo, CCR (center crossing pairs top-bottm) RL, and top string of side triangles coming from top center crossing pair left-right) OO.
So my encoding would be uoou ouuo RL OO.
a) note all 4 protruding loop handles are necessary for tension
b) note all 4 protruding loop handles disentangle when pulled to center of figure
c) center top and bottom crossings are opposite in nature: RL
d) rotating the figure around the horizontal axis as indicated does not result in a different crossing pattern.

If the figure is formed from the rdna opening the top center crossing pair is L and the bottom center crossing pair is RD (LR) which is a distinct different look-alike figure. But again rotating the figure 180 degrees does not result in a different figure.

If the figure is formed from OB the following pattern is produced. Note that all is the same as from the dna openings except for the top and bottom center crossing pairs. This time they are RR.. So when the figure is rotated around the horizontal axis the center crossing pairs are LL, resulting in a different look-alike figure. [THIS, HOWEVER, IS THE FIGURE PRODUCED WHEN THE SUN IS CAUGHT FROM OPENING A, so there are only 2 distinct look-alikes from openings A and B.]


Therefore I conjecture there are only 4 distinct look-alike figures for "stone money" formed from "Catch the Sun".

Then I went on to D'Antoni One.
a) you can alter the resultant figure by weaving with the little finger first. in D'Antoni's description, reverse steps 2 and 3.
2. R1, over R2 loop, picks up R5n. R3, over palm string and R2 loop, picks up lower R1f. Release 1 loops. R1, over R2 loop, and from below removes R3 loop.
3. R5, over R2 loop, picks up R1f. R2, over palm string and R2 loop, picks up lower R5n. Release 5 loops. R5, from below, removes upper R2 loop.
b) you can weave with the off hand (i.e. with left hand for opening A)

My display of the figure for analysis is:

a few explanations are in order:
The top left figure is formed with your directions. i will abbreviate the process as "OA RH 1-5". This means that your moves 2 and 3 are in your order. The figure to the top right is "OA RH 5-1" which indicates the order of these two moves is reversed.

Note that for the bottom left figure "op A LH 1-5" a modification of your procedure is needed between your moves 4 and 5. Tthe R1 loop must be taken off R1 and slipped under palmar string and back onto R1 without twists. All else is the same. This modification will be used in the future when obviously needed.

Note that only two distinct figures are formed. RH 1-5 is same as LH 5-1, and LH 1-5 is same as RH 5-1.

With appropriate tweakings in procedure, look-alikes can be formed from all four basic openings. (A, B, ldna, rdna). let's examines them one by one.
four observations are in order before we proceed.

1) the figures are best distinguished by the top and bottom center crossing pairs. they are RR dominant or LL dominant, as one would expect.
2) both top and bottom center traverses are over-over
3) the protruding loops are top pair catching and bottom releasing.
4) the side triangles both have the string coming from the top center crossing pair on top of its crossing string (OO).

As I sketched the possible DWD's I set myself small rhythms of doing, so I could insure better accuracy. The top and bottom center pattern was done first, then the top transversal and then the bottom. The top crossing patterns of the two side triangles were sketched. Then the final two inkingson the side traiangles were to complete the triangles by using a block move. Whichever way the string was running, the next crossing was a block or a forced-under mode. Block once, block twice, and the sketch was finished. These two locking patterns were absolutely necessary for the integrity of the side corners of the double diamonds.

New figures formed by three rotations are investigated.

1) the figure is rotated around a center horizontal axis as one would do by my "rotating the figure.
2) the figure is rotated around a center vertical axis
3) the figure is spun 180 degrees around a center point
4) the figure may be rotated around vertical axis then around a horizontal axis

Now we enumerate the look-alikes possible by rotating these two figures:

i first took the "op A LH 5-1" figure. It is RR for center crossings, and central transverse strings are over -over. Rotating around horizontal axis results in LL under-under, around vertical axis RR under-under, and around vertical the horizontal LL over-over: 4 distinct figures.

OA LH 5-1 original
OA LH 5-1 rotated around horizontal axis
OA LH 5-1 rotated around vertical axis
OA LH 5-1 rot. around hor. then vert. axis
uoou uoou RR UU
ouuu ouuo LL OO
ouuo ouuo RR OO
uoou uoou LL OO

Then by experimenting with different procedures within the framework of figure D'Antoni One and adding the variants possible with D’Antoni Two, i distinguished 16 different DWD lookalikes:

Uoou Uoou LL OO d'antoni one OA RH 1-5 rot around y axis then rot around x axis
Uoou Uoou LL UU d'antoni one OA RH 5-1(LH 1-5)
Uoou Uoou RR OO d'antoni one OA RH 5-1 rot around y axis then rot around x axis
Uoou Uoou RR UU d'antoni one OA RH 1-5 (LH 5-1)
Ouuo Ouuo LL OO d'antoni one OA RH 5-1 rotation around y axis
Ouuo Ouuo LL UU d'antoni one OA RH 1-5 rotation around x axis
Ouuo Ouuo RR OO d'antoni one OA RH 1-5 rotation around y axis
Ouuo Ouuo RR UU d'antoni one OA RH 5-1 rotation around x axis
The figures derived from D'antoni Two have the following patterns. They answer the question where are the RL and LR figures.

Uoou Uoou LR OO d'antoni two OA RH 1-5 (L2 <<) (center rotation 180)
Uoou Uoou LR UU d'antoni two OA RH 1-5 (L2 <<)
Uoou Uoou RL OO d'antoni two OA RH 5-1 (L2 >>) (center rotation 180)
Uoou Uoou RL UU d'antoni two OA RH 5-1 (L2 >>)
Ouuo Ouuo LR OO d'antoni two OA RH 1-5 (L2<<) rotation around x axis (center rotation 180)
Ouuo Ouuo LR UU d'antoni two OA RH 1-5 (L2 $\ll$ ) rotation around $x$ axis
Ouuo Ouuo RL OO d'antoni two OA RH 5-1 (L2 >>) (center rotation 180)
Ouuo Ouuo RL UU d'antoni two OA RH 5-1 (L2 >>) rotation around x axis (ctr. rotation 180)

Revisiting my "Catch the Sun" patterns shows the following:
Uoou Ouuo LL OO catch the sun OA
Uoou Ouuo RR OO catch the sun OB
Uoou Ouuo LR OO catch the sun rdna
Uoou Ouuo RL OO catch the sun ldna

Ouuo Uoou LL UU catch the sun OB (center rotation 180)
Ouuo Uoou LR UU catch the sun rdna (center rotation 180)
Ouuo Uoou RL UU catch the sun ldna (center rotation 180)
Ouuo Uoou RR UU catch the sun OA (center rotation 180)

This is a list of successful DWD look-alikes.
When I began this investigation as to how many there were possible to make, I was a bit naïve. The problem is tricky, but I think I have a handle on the simpler figures. I call them simpler because they are all made from OA or OB except for the dna "catch the sun".

The upper transverse strings in my framework for the figures are either ouno or uoou (over-under-under-over, under-over-over-under). The details read left to right for the transverse strings are Top, Bottom read left to right. The next two letters are the crossing center pairs read left to right. The L and R stand for the string on top coming from the sides of the figure toward the center. The last two letters denote the orientation of the top string of the side triangles read left to right. OO means the left triangle has the string coming from the top of the figure passes over the lower string of the upper loop, and for the right triangle the string coming from the top of the figure passes over the lower string of the upper loop.

The Lutz and Roth figures (and variants that i came up with) are duplicated in this array of figures. Indeed I argue that these three groups of eight contain all the patterns of the following constrictions.

The catch the sun figures have opposite top and bottom transverse patterns, accentuated by the capitalization of the first letter of the four crossings. i didn't know how to make the figures i intuit as viable (Uoou Ouuo LR UU) or the other 7 similar probabilities. It was only after sketching them that i found the reason they aren't formable on the hands. They are certainly knots and cannot be devolved to the un-knot. The following sketched figures show this.





This first group of sketches contain all of the figures i know how to make on the hands (as indicated) plus several redundancies i have not enumerated. They also are all within the side triangle patterns OO and UU. i then sketched the OU and UO patterns:



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These 64 patterns exhaust the possibilities for the ouvo and uoou pairs of transverse crossings. i manufactured a jig to tie several of the non-string figre produced patterns and they all were knots! i did not exhaustively check every one, but i did check several each of the protruding loop patterns (top pair catching, bottom pair catching, diagonally opposite loops catching)

The ldna and rdna openings presented a bit of a challenge, but i was able to form look-alikes to the DWD by the following maneuver:
for demonstration i chose the ldna opening and modified the loom by putting the R2 loop only up through the R5 loop and returning to position. figure top left below. there is now a new wrinkle to consider ( the under-over of the top center transverse string). note that i tried all variations of index up through outer loops. the resultant was only two distinct figures. i discovered no new figures when i tried all variations of index loops up through and down through outer loops (my ink scribbles in center of diagrams). note only 4 of the 8 resulted in stone money look-alikes.

i went ahead and began to sketch these figures, but in my investigations i became increasingly frustrated with the figures under tension since they tended to lose a "shoulder". The top left figure above loses its top left shoulder when i add two diamonds to the figure. So i discarded the the dna figures i at first thought were look-alikes.

Then I turned my attention to adding the ouou and uouo patterns to my investigation. I found to my surprise that the ouou and uouo patterns limited the variables for the top and bottom center crossing pairs. The formed a triangle much like the side triangles and were similarly constrained to only one possible crossing pattern depending on whether the left or the right string was dominant. The resulting 80 figures follow:




$$
\left(\mathrm{Ind}_{\mathrm{Gl}}\right)^{4}
$$



Ouv LN


$$
\binom{\mathrm{zel}}{\mathrm{ge}} 5
$$



All the above have the side triangle patterns OO and UU.

The following 40 figures have the side triangle patterns OU and UO .


$$
\left(\begin{array}{c}
4 \mathrm{u} \text { set } \\
(2)
\end{array}\right.
$$






I then collated the different groups of patterns and found:
group 1:
the transverse strings were either ouuo or uoou, the center crossing pairs were RR, RL,LR,LL, the side triangles top strings were OO,UU.
since the ouuo nd the uoou transverse realities permitted all four center crossing pairs to form figures there were 32 figures ( 24 of which were string figures).
the 32 figures can be sorted as follows:
8 with no protruding loops catching when pulled to center of figure (sf's)
8 with the bottom two protruding loops catching (sf's)
8 with the top two protruding loops catching (sf's)
8 with all protruding loops catching (definitely not the unknot)
group 3:
the same as group 1 except the side triangle strings were OU,UO
8 with diagonal pair bottom left to top right catching
8 with diagonal pair top left to bottom right catching
8 with both right catching
8 with both left catching
i checked several of each variety with a small jig i set up to form the figures with a string with two ends that $i$ tied when the figure was formed. all that $i$ checked were a knot.
group 2
transverse pairs with ouou or uouo paired with ouuo. also transverse pairs with ouou or uouo paired with uoou. the center crossing pairs for the uoou or the ouno transverse could be any center crossing pattern, but the center crossing pattern for the ouou can only be L and the center crossing pattern for the uouo can only be R. the reason is interesting. the ouuo and uoou pattern works only when the tension of the tranverse string is out from the center of the figure. the side triangles are restricted because the tension is toward the center and only one set up works with tension in that direction. similarly the ouou and uouo transverse patterns work only one way with the center crossing pair because of the ou and uo center crossings of the transverse string (oUOu, uOUo ). The side triangle pairs were OO and UU

2 with diagonal pair bottom left to top right catching
2 with diagonal pair top left to bottom right catching
2 with both right catching
2 with both left catching
4 with 3 catching loops and the free loop top right
4 with 3 catching loops and the free loop top left
4 with 3 catching loops and the free loop bottom right
4 with 3 catching loops and the free loop bottom left
4 with one catching loop top right
4 with one catching loop top left

4 with one catching loop bottom right
4 with one catching loop bottom left
again i checked several figures and all were knots.
group 4
the same as group 2 except side triangle pairs OU and UO
2 with no loops catching (BOTH KNOTS!!)
2 with two top loops catching (BOTH KNOTS!!)
2 with two bottom loops catching (BOTH KNOTS!!)
2 with all loops catching
4 with 3 catching loops and the free loop top right
4 with 3 catching loops and the free loop top left
4 with 3 catching loops and the free loop bottom right
4 with 3 catching loops and the free loop bottom left
4 with one catching loop top right
4 with one catching loop top left
4 with one catching loop bottom right
4 with one catching loop bottom left
again i checked several figures and all were knots.
it is interesting how we both missed the fact that the top crossing pairs actually form a triangle similar to the side triangles which limits the number of different configurations when the interior transverse string crossings of the center triangles are OU and UO.

So I conclude there are only 144 look-alike DWD's that maintain themselves under tension as I understand this crucial definition.

There are only 24 look-alike DWD's which are the un-knot (i.e. authentic string figure possibilities)

I note here that the issue is still not decided. Mine is only a conjecture. Note D'Antoni's following discussion:

A picture is worth a thousand words. Here is how I determined the number of stable DWD's, under tension. The number in parenthesis represents how many unique crossings can appear at that point in the figure without destroying the DWD.


Out of $2^{16}$ look-alike figures, only $2^{8}$ survive under tension.

I might add here that subsequents investigations of "tension" led me to the following:


Catch the sun OA two diamonds center


D'antoni one RH 1-5 two diamonds center

Clearly these are different figures after wrestling into form.


but they are symmetric feeling even with my uneven hands. your halloween drawing is below.



It looks similar to your initial figure one as it should, but it torques under tension as below.


so the top left shoulder and the bottom right shoulder are twisting around the transverse. I think we have to define "under tension" and what a genuine "look-alike under tension" really means.
i had used the two diamonds test for many of the figures i was making, and for the oouu and uuoo transverse patterns i always lost shoulders; so i discarded them in my counting. i took only the catch the sun, d'antoni one, and d'antoni two series as legitimate look-alikes.

